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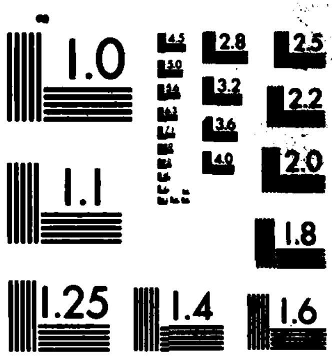
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ABSTRACT

In this note, we derive the asymptotic distribution of logarithm of the likelihood ratio statistic for testing the hypothesis that the number of signals is equal to q against the alternative that it is equal to k (specified) for a special case. This distribution is not chi-square. The above statistic also arises (see Wax and Kailath (1985)) in studying consistency property of MDL and AIC criteria for detection of the number of signals.

1. INTRODUCTION

In the area of signal processing, a model that is often used involves modeling the observation (complex) vector as a linear combination of the elements of the signal vector and vector of white noise. Under this model, the (unknown) number of signals is related to the multiplicity of the smallest eigenvalue of the covariance matrix of the observation vector. Recently, Wax and Kailath (1985) used Akaike's AIC criterion and Schwartz-Rissanen's MDL criterion for determination of the number of signals and stated that the AIC criterion is not consistent whereas the MDL criterion is consistent. Parts of the proofs of the above statements are based upon the assumption that the asymptotic distribution of $-2 \log L_{tk}$ is chi-square where L_{tk} denotes the likelihood ratio test statistic for testing H_t against H_k where t and $k(k > t)$ are specified. In this note, we derive the asymptotic distribution of the above test statistic for trivariate case and point out that it is not chi-square. We will also derive analogous result when the underlying distribution is real multivariate normal.

2. SOME CRITERIA FOR DETECTION OF SIGNALS

Consider the model

$$\underset{\sim}{x}(t) = A\underset{\sim}{s}(t) + \underset{\sim}{n}(t) \tag{2.1}$$

where $A = [A(\underset{\sim}{\Phi}_1),\ldots,A(\underset{\sim}{\Phi}_q)]$, $s(t) = (s_1(t),\ldots,s_q(t))'$, $s_i(t)$ is complex waveform associated with i-th signal, $A(\underset{\sim}{\Phi}_i)$ is a complex vector which depends upon the unknown vector associated with i-th signal, $\underset{\sim}{n}(t)$ is noise vector. Also, $\underset{\sim}{s}(t)$ and $\underset{\sim}{n}(t)$ are distributed independently as complex multivariate normal with $E(\underset{\sim}{s}(t)) = \underset{\sim}{0}$, $E(\underset{\sim}{n}(t)) = \underset{\sim}{0}$, $E(\underset{\sim}{s}(t)\overline{\underset{\sim}{s}(t)}') = \Psi$ and $E(\underset{\sim}{n}(t)\overline{\underset{\sim}{n}(t)}') = \sigma^2 I_p$, where σ^2 is unknown. Here $\underset{\sim}{y}'$ and $\overline{y}$ respectively denote the transpose and conjugate of $\underset{\sim}{y}$. The number of signals q is unknown. Also, $\underset{\sim}{x}(t_1)\ldots$, $\underset{\sim}{x}(t_n)$ is a sample from a complex multivariate normal population with mean vector $\underset{\sim}{0}$ and covariance matrix Σ_2. Now, let M_t denote the t-th model which states that the number of signals is t. Wax and Kailath (1985) proposed using the AIC criterion and MDL criterion for the selection of the number of signals. According to the AIC criterion, we select the model for which

$$AIC(t) = -2 \log L_t + \nu(t,p) \tag{2.3}$$

is minimum where $\nu(t,p) = t(2p - t) + 1$. According to the MDL criterion, we select the model for which

$$MDL(t) = - \log L_t + \nu(t,p)\frac{\log n}{2} \tag{2.4}$$

is minimum. Now, let H_t denote the hypothesis that the number of signals is equal to t. The logarithm of the likelihood ratio test statistic for testing H_r against H_k for specified values of r and k ($r < k$) is known to be $\log L_r - \log L_k$ where

$$\log L_t = -n\{\sum_{i=t+1}^{p} \log \ell_i - \log(\sum_{i=t+1}^{p} \ell_i/(p-t))\}. \quad (2.5)$$

Wax and Kailath (1985) pointed out that the MDL criterion is consistent whereas the AIC criterion is not consistent. In the proofs of the above statements, they have incorrectly assumed that $-2 \log L_{rk}$ is distributed asymptotically as chi-square.

Now let $\underset{\sim}{\Theta}$ denote the parametric space. Under H_t, the supremum of the logarithm of the likelihood function, $L(\underset{\sim}{\Theta})$, might be reached at a boundary point of $\underset{\sim}{\Theta}$ and not in any inner point; such a point is $\lambda_1 = \ell_1, \ldots, \lambda_k = \ell_k$, $\lambda_{t+1} = \ldots = \lambda_p = \sum_{j=t+1}^{p} \ell_j/(p-t)$. So the conditions for $-2 \log L_{rk}$ to be distributed as chi-square asymptotically are not satisfied. In Section 3, we derive the asymptotic distribution of $-2 \log L_{rk}$ for the special case when $p = 3$. Analogous result is also derived when $\underset{\sim}{x}(t)$ is distributed as real multivariate normal with mean vector $\underset{\sim}{0}$ and covariance matrix Σ_2.

3. DISTRIBUTIONS OF THE LRT STATISTICS FOR DETECTION OF NUMBER OF SIGNALS

In this section, we will derive the asymptotic distribution of $-2 \log L_{rk}$ when $p = 3$, $r = 0$ and $k = 1$. The following definition of complex Gaussian matrix is needed in the sequel.

Let $A = (a_{j\ell}) = R + iS$, where A: pxp is a Hermitian random matrix, $R = (r_{j\ell})$ and $S = (s_{j\ell})$. Assume $s_{jj} = 0$, and the distinct elements of R and the upper-diagonal elements of S are independent real normal variables. Also, we assume that the variances of the off-diagonal elements of R and S are equal to 1 and the variances of the diagonal elements of R are equal to 2. Then, $A = R + iS$ is known (see Krishnaiah (1976)) to be the central or noncentral complex Gaussian matrix accordingly as $E(A) = 0$ or $E(A) \neq 0$.

Now, let $\underset{\sim}{x}_1, \ldots, \underset{\sim}{x}_n$ be distributed independently as complex trivariate normal with mean vector $\underset{\sim}{0}$ and covariance matrix I_3 where I_3 is an identity matrix of order 3x3. Also, let $\ell_1 \geq \ell_2 \geq \ell_3$ be the roots of the equation.

$$\left|\frac{1}{n}\sum_{j=1}^{n} \underset{\sim}{x}_j \bar{\underset{\sim}{x}}_j' - \ell I_3\right| = 0. \tag{3.1}$$

Rewrite this equation as

$$\left|\sqrt{2n}\left(\frac{1}{n}\sum_{j=1}^{n} \underset{\sim}{x}_j \bar{\underset{\sim}{x}}_j' - I_3\right) - \sqrt{2n}(\ell - 1)I_3\right| = 0. \tag{3.2}$$

By the central limit theorem, we have

$$\mathcal{L}\left(\sqrt{2n}\left(\frac{1}{n}\sum_{j=1}^{n} \underset{\sim}{x}_j \bar{\underset{\sim}{x}}_j' - I_3\right)\right) \to \mathcal{L}(A_3) \tag{3.3}$$

as $n \to \infty$ where A_3 is 3x3 central complex Gaussian matrix. So the asymptotic distribution of $\sqrt{2n}(\ell_1-1, \ell_2-1, \ell_3-1)$ is the same as distribution of the eigenvalues $\tau_1 \geq \tau_2 \geq \tau_3$ of A_3. It is well known (see Wigner (1965)) that the joint density of (τ_1, τ_2, τ_3) is

$$h(t_1,t_2,t_3) = C\{\prod_{1\leq j<m\leq 3}(t_j-t_m)^2\}\exp(-\sum_{j=1}^{3} t_i^2/4), \qquad (3.4)$$
$$\infty > t_1 \geq t_2 \geq t_3 > -\infty.$$

where

$$C = 2^{-7}\pi^{-3/2}.$$

Since $\lim_{n\to\infty}(\ell_1-1) = 0$ a.s., we can use Taylor's expansion and get from (2.5),

$$\begin{aligned} &- 2[\sup_{\underset{\sim}{\theta}\in\underset{\sim}{\Theta}_1} L(\underset{\sim}{\theta}) - \sup_{\underset{\sim}{\theta}\in\underset{\sim}{\Theta}_0} L(\underset{\sim}{\theta})] \\ &= [n(\ell_1-1)^2 + \frac{n}{2}(\sum_{j=2}^{3}(\ell_i-1))^2 - \frac{n}{3}(\sum_{j=1}^{3}(\ell_i-1))^2](1+o(1)) \text{ a.s.} \end{aligned} \qquad (3.5)$$

Here, Θ_k denotes the parameter space under H_k.

Write

$$W_n = - 2[\sup_{\underset{\sim}{\theta}\in\underset{\sim}{\Theta}_1} L(\underset{\sim}{\theta}) - \sup_{\underset{\sim}{\theta}\in\underset{\sim}{\Theta}_0} L(\underset{\sim}{\theta})].$$

Then

$$W_n \xrightarrow{\mathcal{D}} \frac{1}{2}\tau_1^2 + \frac{1}{4}(\tau_2+\tau_3)^2 - \frac{1}{6}(\tau_1+\tau_2+\tau_3)^2, \qquad (3.6)$$

as $n \to \infty$. Now, let $\eta_1 = \frac{1}{\sqrt{2}}\tau_1$, $\eta_2 = \frac{1}{\sqrt{2}}\tau_2, \eta_3 = \frac{1}{\sqrt{2}}\tau_3$. Then

$$W_n \xrightarrow{\mathcal{D}} \eta_1^2 + \frac{1}{2}(\eta_2+\eta_3)^2 - \frac{1}{3}(\eta_1+\eta_2+\eta_3)^2 \overset{\Delta}{=} W, \qquad (3.7)$$

where the joint density of (η_1, η_2, η_3) is

$$g(\eta_1, \eta_2, \eta_3)$$

$$= 2^{-5/2}\pi^{-3/2}\exp(-\tfrac{1}{2}(\eta_1^2+\eta_2^2+\eta_3^2))\prod_{1\le j<m\le 3}(\eta_j-\eta_m)^2. \tag{3.8}$$

If we write

$$\begin{pmatrix} \eta_1 \\ \eta_2 \\ \eta_3 \end{pmatrix} = \begin{pmatrix} \frac{2}{\sqrt{6}} & 0 & \frac{1}{\sqrt{3}} \\ -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{3}} \\ -\frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{3}} \end{pmatrix} \begin{pmatrix} Y_1 \\ Y_2 \\ Y_3 \end{pmatrix},$$

then $\eta_1 - \eta_2 = \sqrt{\frac{3}{2}}\, Y_1 - \sqrt{\frac{1}{2}}\, Y_2$, $\eta_2 - \eta_3 = \sqrt{2}\, Y_2$,

$$\eta_1 - \eta_3 = \sqrt{\tfrac{3}{2}}\, Y_1 + \frac{1}{\sqrt{2}}\, Y_2,$$

and

$$W = [\frac{2}{\sqrt{6}}\eta_1 - \sqrt{\frac{1}{6}}(\eta_2+\eta_3)]^2 = Y_1^2. \tag{3.9}$$

Thus the joint density of (Y_1, Y_2, Y_3) is

$$f(y_1, y_2, y_3) = 2^{-7/2}\pi^{-3/2}(3y_1^2-y_2^2)^2 y_2^2 \exp[-\tfrac{1}{2}(y_1^2+y_2^2+y_3^2)]$$

$$\sqrt{3}y_1 > y_2 > 0,\ -\infty < y_3 < \infty. \tag{3.10}$$

Hence, the density of Y_1 is

$$f_1(y_1) = 2^{-3}\pi^{-1}\int_0^{\sqrt{3}y_1}(3y_1^2-y_2^2)^2 y_2^2 \exp[-\tfrac{1}{2}(y_1^2+y_2^2)]\,dy_2, \quad y_1 > 0. \tag{3.11}$$

So the density of Y_1^2 is

$$a(u) = \frac{1}{2} u^{-1/2} f_1(\sqrt{u}) = 2^{-4}\pi^{-1}u^{-1/2}e^{-u/2}\int_0^{\sqrt{3u}} (3u-y^2)^2 y^2 e^{-y^2/2}\,dy, \quad u > 0. \tag{3.12}$$

Now, let

$$J_k(x) = \int_0^x y^{2k} e^{-y^2/2}\,dy, \quad k = 0,1,2,3.$$

Then we have

$$J_k(x) = - x^{2k-1}e^{-x^2/2} + 2(k-1)J_{k-1}(x) \tag{3.13}$$

for $k = 1,2,3$. Thus,

$$\begin{aligned} J_3(x) &= (-x^5-5x^3-15x)e^{-x^2/2} + 15J_0(x), \\ J_2(x) &= (-x^3-3x)e^{-x^2/2} + 3J_0(x), \\ J_1(x) &= -xe^{-x^2/2} + J_0(x). \end{aligned} \tag{3.14}$$

So, we have

$$\begin{aligned} a(u) &= \frac{1}{16\pi} u^{-1/2}e^{-u/2}[J_3(\sqrt{3u}) - 6uJ_2(\sqrt{3u}) + 9u^2J_1(\sqrt{3u})] \\ &= \frac{1}{16\pi} u^{-1/2}e^{-u/2}[((3u)^{3/2}-15\sqrt{3u})e^{-3u/2} \\ &\quad + (9u^2-18u+15)J_0(\sqrt{3u})] \end{aligned} \tag{3.15}$$

$$= \frac{1}{16\pi} (3\sqrt{3u}-15\sqrt{3})e^{-2u} + \frac{1}{16\pi} u^{-1/2}e^{-u/2}(9u^2-18u+15)J_0(\sqrt{3u}),$$

$$(u > 0).$$

where

$$J_o(\sqrt{3u}) = \int_o^{\sqrt{3u}} e^{-y^2/2} dy. \qquad (3.16)$$

From (3.7), (3.9) and (3.15), it follows that the asymptotic distribution of $W_n = -2[\sup_{\underset{\sim}{\theta} \in \Theta_1} L(\underset{\sim}{\theta}) - \sup_{\underset{\sim}{\theta} \in \Theta_o} L(\underset{\sim}{\theta})]$ is not chi-square.

In view of the above counter example, the strong consistency of the MDL criterion does not follow from the argument of Wax and Kailath (1985). But, it follows from the results given in a companion paper by Zhao, Krishnaiah and Bai (1985).

Now we suppose that $x_1,\ldots,x_n$ are px1 i.i.d. real normal random vectors with $E\underset{\sim}{x}_1 = 0$ and $E\underset{\sim}{x}_1\underset{\sim}{x}_1' = \Sigma > 0$. We point out that, the asymptotic distribution of $-2[\sup_{\underset{\sim}{\theta} \in \underset{\sim}{\Theta}_k} L(\theta) - \sup_{\underset{\sim}{\theta} \in \underset{\sim}{\Theta}_q} L(\theta)]$ here also is not chi-square.

Now let $R = (r_{jm})$: pxp be a symmetric random matrix. We assume that $\{r_{jm};\ j \leq m = 1,\ldots,p\}$ are independent normal variables with means zero, $var(r_{jj}) = 1$ and $var(r_{j\ell}) = 1/2$ for all $j < \ell$. Then R is known to be distributed as (real) central or noncentral Gaussian matrix accordingly as $E(R) = 0$ or not.

Let $\ell_1 \geq \ell_2 \geq \ell_3$ be the eigenvalues of $\frac{1}{n}\sum_{j=1}^{n}\underset{\sim}{x}_j\underset{\sim}{x}_j'$ and $\tau_1 \geq \tau_2 \geq \tau_3$ be the eigenvalues of 3x3 central (real) Gaussian matrix. Then, using the same argument as before, the joint distribution of $\sqrt{\frac{n}{2}}(\ell_1-1,\ell_2-1,\ell_3-1)$ tends to that of (τ_1,τ_2,τ_3). It is well known (e.g., see Anderson (1984)) that the joint density of (τ_1,τ_2,τ_3) is given by

$$h(t_1,t_2,t_3) = \frac{1}{\sqrt{2\pi}}\exp\{-\frac{1}{2}\sum_{j=1}^{3} t_j^2\}\prod_{1\le i\le j\le 3}(t_i-t_j) \tag{3.17}$$

$$(\infty > t_1 > t_2 > t_3 > -\infty).$$

Let $Y_1 = \frac{2}{\sqrt{6}}\tau_1 - \frac{1}{\sqrt{6}}(\tau_2+\tau_3)$, $Y_2 = \frac{1}{\sqrt{2}}(\tau_2-\tau_3)$, $Y_3 = \frac{1}{\sqrt{3}}(\tau_1+\tau_2+\tau_3)$. Then the joint density of (Y_1,Y_2,Y_3) is given by

$$f(y_1,y_2,y_3) = \frac{1}{2\pi}(3y_1^2-y_2^2)y_2\exp\{-\frac{1}{2}(y_1^2+y_2^2+y_3^2)\}, \tag{3.18}$$

$$\sqrt{3}y_1 > y_2 > 0,\ -\infty < y_3 < \infty.$$

Using the same argument, we have

$$W_n \xrightarrow{\mathcal{L}} W = Y_1^2. \tag{3.19}$$

By (3.18), the density of Y_1 is

$$\begin{aligned} f_1(y_1) &= \frac{1}{\sqrt{2\pi}}\int_0^{\sqrt{3}y_1}(3y_1^2-y_2^2)y_2\exp\{-\frac{1}{2}(y_1^2+y_2^2)\}dy_2 \\ &= \frac{1}{\sqrt{2\pi}}(3y_1^2-2)e^{-y_1^2/2} + \frac{2}{\sqrt{2\pi}}e^{-2y_1^2},\quad y_1 > 0. \end{aligned} \tag{3.20}$$

Thus, the density of W is

$$\frac{1}{2}u^{-1/2}f_1(\sqrt{u}) = \frac{1}{2\sqrt{2\pi}}(3u^{1/2}-2u^{-1/2})e^{-u/2} + \frac{1}{\sqrt{2\pi}}u^{-1/2}e^{-2u}, \tag{3.21}$$

$$0 < u < \infty.$$

From (3.19) and (3.21), it follows that the asymptotic distribution of $W_n = -2[\sup_{\underset{\sim}{\theta}\in\underset{\sim}{\Theta}_1} L(\underset{\sim}{\theta}) - \sup_{\underset{\sim}{\theta}\in\underset{\sim}{\Theta}_0} L(\underset{\sim}{\theta})]$ is not chi-square distribution.

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